Problem Sheet 4

Problem 1

For a ring A and a prime ideal $\mathfrak{p} \subseteq A$, we denote by $A_{\mathfrak{p}} := A[(A \setminus \mathfrak{p})^{-1}]$ its localization at $\mathfrak{p}$.

Show that a noetherian integral domain A is Dedekind if and only if every localization $A_{\mathfrak{p}}$, $\mathfrak{p} \neq (0)$, is a principal ideal domain.

Problem 2

Let k be an algebraically closed field with char $k \neq 2$. Consider the rings

$$A = k[X] \subseteq B = k[X,Y]/(Y^2 - X^3 + X).$$

(a) Show that A and B are Dedekind domains.

(b) Prove that $X, Y \in B$ are irreducible but not prime. (In particular, B is not a PID.)

(c) Determine all prime ideals of A that are ramified in B.

Problem 3

Determine the prime ideal factorization of $(455) \subseteq \mathbb{Z}[\sqrt{-6}]$. Compute the order of every occurring prime ideal in the class group $Cl_{\mathbb{Q}(\sqrt{-6})}$.

Problem 4

Consider the fields $\mathbb{Q}(\zeta_{23})$ and $K = \mathbb{Q}(\sqrt{-23}) \subseteq \mathbb{Q}(\zeta_{23})$. Let $\mathfrak{p}$ denote the prime ideal $(2, (1+\sqrt{-23})/2)$ of $\mathcal{O}_K$. Show that there exists a unique prime ideal $\mathfrak{P}$ of $\mathbb{Q}(\zeta_{23})$ above $\mathfrak{p}$ and that $\mathfrak{P}$ is not a principal ideal.